Lecture 1. The Kakutani–Glicksberg–Ky Fan fixed point theorem

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Theorem 1 (Kakutani–Glicksberg–Ky Fan Fixed Point Theorem). *Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X. Let $\Gamma\colon K \rightrightarrows K$ be a correspondence with closed graph and nonempty convex values. Then there exists a point x in K such that $x \in \Gamma(x)$.*

Proof. Let N be a closed symmetric neighborhood and define $F_N = \{x \in X : x \in (\Gamma(x) + N) \cap K\}$. We first prove that F_N is closed. The correspondence defined by $N(x) = N + x$ has closed graph, and so the correspondence defined by $M(x) = N(x) \cap K$ is upper hemicontinuous, and so is the correspondence defined by $M \circ \Gamma(x) = (\Gamma(x) + N) \cap K$. Hence, the correspondence defined by $(\mathrm{id} \cap (M \circ \Gamma))(x) = \{x\} \cap (\Gamma(x) + N) \cap K$ is also upper hemicontinuous. The set $(\mathrm{id} \cap (M \circ \Gamma))^u(\emptyset)$ is thus open. Since $F_N = X \setminus (\mathrm{id} \cap (M \circ \Gamma))^u(\emptyset)$, we have that F_N is closed. We now prove that F_N is nonempty. By compactness of K, there exist $a_0, a_1, \ldots, a_d \in K$ such that $K \subseteq \bigcup_{i=0}^{d}(a_i + N)$. Define the convex hull $S^d = \mathrm{conv}(\{a_0, a_1, \ldots, a_n\})$, which is compact with dimension at most d, and so can be identified with a subset of $\mathbb{R}^d$. Consider the correspondence $B\colon S^d \rightrightarrows S^{d-1}$ defined by $B(x) = (\Gamma(x) + N) \cap S^d$. For each $x \in S^d$, the set $B(x)$ is convex since $\Gamma(x)$ and N are convex. For each $x \in S^d$, we have $\Gamma(x) \subseteq K \subseteq (S^d + N)$, hence $\Gamma(x) \cap (S^d + N) \neq \emptyset$. Fix $x \in \mathbb{R}^d$ and consider $u \in \Gamma(x) \cap (S^d + N)$ with $u = v + w$ where $v \in S^d$ and $w \in N$. Since N is symmetric, $-w \in N$, and so $v = u - w \in S^d \cap (\Gamma(x) + N)$, hence $S^d \cap (\Gamma(x) + N) \neq \emptyset$. The conditions of Kakutani's theorem are satisfied by B (with the identification of S^d with some subset of $\mathbb{R}^d$), and so there exists a point $x_0 \in S^d$ such that $x_0 \in S^d \cap (\Gamma(x_0) + N)$. Hence, $x_0 \in K \cap (\Gamma(x_0) + N)$, and so $F_N \neq \emptyset$. We conclude the proof by compactness. If N and M are closed symmetric neighborhoods, then $N \cap M$ is symmetric and $F_{N \cap M} \subseteq F_N \cap F_M$. Therefore any finite subcollections of sets F_N has nonempty intersection, which implies by compactness of K that $\bigcap_N F_N \neq \emptyset$. Let $x_0 \in \bigcap_N F_N$. If $x_0 \notin \Gamma(x_0)$, then there is a symmetric neighborhood M (e.g., $M = \{0\}$) such that $x_0 \notin F_M$: a contradiction. Therefore, $x_0 \in \Gamma(x_0)$. □

Reference. We follow Berge (1963) which is a standard. The proof extends Kakutani's fixed point theorem in $\mathbb{R}^d$ to locally convex space by using the fact that compact sets in infinite-dimensional spaces do not behave so differently than finite-dimensional subsets. Another method of proof relies on some fixed-point theorem of Halpern and Bergman and can be found in Aliprantis&Border (2006).

Remark. At the basis of many fixed point theorems lies a number of equivalent results: Sperner's lemma, the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma, and the

Brouwer's fixed point theorem (a combinatorial result, a set-covering result, and an algebraic topology result; each result can be proved separately using different arguments, but each implies the other). From our purpose, we start with Sperner's lemma and use it to prove the KKM lemma. We then use the KKM lemma to prove Kakutani's theorem in $\mathbb{R}^d$ and use an approximation argument for compact sets in infinite dimensional spaces to prove the result in this setting. This is exactly the route followed by Berge (1963) p.171-174&251. Other standard routes exist (in a more or less combinatorial fashion), but they all build on the same first step which is to use any of the three equivalent results: Sperner's lemma, the KKM lemma, or the Brouwer's fixed point theorem. See also Appendix B in Aubin (1979), S.4-6. in Border (1985), S.17.8-9. in Aliprantis&Border (2006), or S.2.5.9.D. in Granas&Dugundji (2003).

Remark. By the closed graph theorem and the fact that closed sets in compact space are compact, the theorem is equivalently valid for upper hemicontinuous correspondences with nonempty convex closed values or upper hemicontinuous correspondences with nonempty convex compacts values. Note that Berge (1963) requires compact values in the definition of upper hemicontinuity.

Lemma 2 (Sperner's Lemma). *Let T be a triangulisation of a d-simplex Δ^d. Suppose the vertexes in T are labeled by elements of $\{0, 1, \ldots, d\}$ such that*[1]*:*

1. each vertex of Δ^d has a distinct label;

2. the label of any vertex in T that lies on a facet of Δ^d matches one of the labels of the facet of that same facet.

Then there exists an odd number of d-simplices $\sigma \in T$ such that each of the vertices of σ take a distinct label. In particular, there exists one such simplex in the triangulation.

Proof. The proof works by induction on d. If $d = 0$, the result is trivially true. Let $d > 0$ and suppose the result is true for $d - 1$. Let T be a triangulation of Δ^d with Sperner labeling. For each $\sigma \in T$, define $F(\sigma)$ as the number of $(d-1)$-simplices labeled $(0, 1, \ldots, d-1)$ among the $d + 1$ facets of σ. We now compute $S = \sum_{\sigma \in T} F(\sigma)$ in two ways. For the first count, note that the $(d + 1)$-simplices appearing in S exclusively come from the A d-simplices of T labeled $(0, 1, \ldots, d - 1, h)$ with $h \in \{0, 1, \ldots, d - 1\}$ and from the B complete d-simplices of T. Each of the A d-simplices have 2 faces labeled $(0, 1, \ldots, d - 1)$ whereas each of the B complete d-simplices have only 1 face labeled $(0, 1, \ldots, d-1)$. Thus, $S = 2A+B$. For the second count, note that the C $(d-1)$-simplices labeled $(0, 1, \ldots, d - 1)$ on the boundary of Δ^d only contributes 1 each to S whereas the D $(d - 1)$-simplices labeled $(0, 1, \ldots, d - 1)$ in the interior of Δ^d contributes 2 each to S since each such $(d - 1)$-simplex in the interior is the face of 2 d-simplices in T. Thus, $S = 2A + B = C + 2D$. Now, the C $(d - 1)$-simplices labeled $(0, 1, \ldots, d - 1)$ on the boundary of Δ^d necessarily come from the facet of Δ^d labeled $(0, 1, \ldots, d - 1)$ and so are the complete simplices in a Sperner labeling of a triangulation of that facet. By the induction hypothesis, C is odd, so B is odd. □

Reference. This combinatorial proof by induction is very standard. Another proof by induction in the language of graph theory exists (see Meunier). Proofs from the algebraic topology side, inspired by the proof of Brouwer's fixed point, also exist (see discussion in McLennan&Tourky (2008)); but they can actually be interpreted as proving Brouwer's

[1]A label of this form is called a Sperner labeling. Note that this is a boundary condition: the conditions only apply to the vertex lying in the facets of Δ^d; the inside vertexes in T can be labeled arbitrarily.

theorem using tools from algebraic topology and using it to derive Sperner's lemma. The essence of Sperner's lemma is combinatorial and so we use combinatorial arguments to prove it. Even if this proof is not constructive, there exist some which give an explicit algorithm for locating the completely labeled cells in a triangulated simplex.

Lemma 3 (Lebesgue's Covering Theorem)**.** *Let $F_1, \ldots, F_d$ be a closed finite covering of a compact metric space X. Then there exists a number $\varepsilon > 0$ such that for each set A with* $\operatorname{diam}(A) < \varepsilon$*, we have $\bigcap\{F_i : F_i \cap A \neq \emptyset\} \neq \emptyset$.*

Proof. We first prove that: if $F_1, \ldots, F_d$ are closed nonempty sets in a compact metric space X with $F_1 \cap \cdots \cap F_d = \emptyset$, then there exists $\varepsilon > 0$ such that $(\bigcap_{i=1}^d F_i) \cap A \neq \emptyset$ implies $\operatorname{diam}(A) \geq \varepsilon$. Let Z be the compact metric space $F_1 \times \cdots \times F_n$ and consider the continuous map $f : Z \longrightarrow \mathbb{R}$ defined by $(x_1, \ldots, x_d) \mapsto \max\{d(x_i, x_j) : 1 \leq i \leq j \leq d\}$. Because $F_1 \cap \cdots \cap F_d = \emptyset$, the function f never reaches 0 and it thus admits a minimum $\varepsilon > 0$. If $A \subseteq X$ is such that $(\bigcap_{i=1}^d F_i) \cap A \neq \emptyset$, there is $x'_i \in A \cap F_i$ for each i. Since $f(x'_1, \ldots, x'_d) \geq \varepsilon$, then $d(x'_i, x'_j) \geq \varepsilon$ for at least some i and j, and so $\operatorname{diam}(A) \geq \varepsilon$ by definition. The result then follows directly by contradiction. □

Reference. We follow Granas&Dujungdi (2003) p.90. Another proof is in Berge (1963) p.95. The preliminary result in the proof is often known as Lebesgue's number lemma and often occurs is its dual form: if $\{U_i : i \in I\}$ is an open cover of a compact metric space X, then there exists $\varepsilon > 0$ such that every set $A \subseteq X$ with $\operatorname{diam}(A) < \varepsilon$ is contained in some U_i.

Lemma 4 (**Knaster–Kuratowski–Mazurkiewicz Lemma**)**.** *Let $\{x_0, x_1, \ldots, x_d\} \subseteq \mathbb{R}^{d+1}$ and $\Delta^d = \operatorname{conv}\{x_i : i \in \{0, 1, \ldots, d\}\}$ the d-simplex with vertices $\{x_0, x_1, \ldots, x_n\}$. Let $F_0, F_1, \ldots, F_d$ be closed subsets of Δ^d such that for every $I \subseteq \{0, 1, \ldots, d\}$, we have $\operatorname{conv}\{x_i : i \in I\} \subseteq \bigcup_{i \in I} F_i$. Then the intersection $\bigcap_{i=0}^d F_i$ is nonempty and compact.*

Proof. The collection of sets $\{F_0, F_1, \ldots, F_d\}$ forms a closed covering of the set Δ^d which is compact by the Heine–Borel theorem. By Lebesgue's covering theorem, there exists $\varepsilon > 0$ such that for each set A in Δ^d with $\operatorname{diam}(A) < \varepsilon$, we have $\bigcap\{F_i : F_i \cap A \neq \emptyset\} \neq \emptyset$. Let T be a triangulation of Δ^d such that $\operatorname{diam}(\sigma) < \varepsilon$ for each $\sigma \in T$. (It is always possible to find such a triangulation). For any vertex v of T, consider the lowest dimensional face $\operatorname{conv}(\{x_{i_0}, \ldots, x_{i_s}\})$ of Δ^d containing v. By assumption, there is $i \in \{i_0, \ldots, i_s\}$ such that $v \in F_i$. If we label v by i and repeat the process for each $v \in V$, we obtain a Sperner labeling for T. Then, by Sperner's lemma, there is a d-simplex $\sigma' \in T$ with vertices $\{v_0, v_1, \ldots, v_d\}$ such that $v_i \in F_i$ for all $i \in \{0, 1, \ldots, d\}$. Hence, $\sigma' \cap F_i \neq 0$ for all $i \in \{0, 1, \ldots, d\}$. Since $\operatorname{diam}(\sigma') < \varepsilon$, we have $\bigcap_{i=0}^d F_i \neq \emptyset$. □

Reference. The proof is standard and follows Berge (1963). Another proof using the sequential characterization of compactness (which is generally proved using Lebesgue's covering theorem) is in Aubin (1979). (A similar proof as in Aubin (1979) is in Border (1985) but is incomplete as such, for the convergence to a same element is not explicitly guaranteed.)

Lemma 5 (**Kakutani's Fixed Point Theorem**)**.** *Let K be a nonempty compact convex subset of $\mathbb{R}^d$. Let $\Gamma : K \rightrightarrows K$ be a correspondence with closed graph and nonempty convex values. Then there exists a point x in K such that $x \in \Gamma(x)$.*

Proof. The result is proved in three steps: (1) for $K = \operatorname{conv}(\{a_0, a_1, \dots, a_d\})$ a d-simplex and Γ a single-valued mapping; (2) for $K = \operatorname{conv}(\{a_0, a_1, \dots, a_d\})$ a d-simplex and Γ a correspondence; (3) for K a nonempty compact convex set.

(1) Every d-simplex has an internal point which can be taken to be the origin 0 without loss of generality. Consider the convex cone $C_i = \{\lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_d a_d : \lambda_0, \dots, \lambda_d \geq 0\}$. Let $B_\lambda(0)$ the closed ball of center 0 and radius λ chosen sufficiently large so that $K \subseteq B_\lambda(x)$ for all $x \in K$. The set $C_i' = C_i \cap B_\lambda(0)$ is compact, hence the correspondence Γ_i defined by $\Gamma_i(x) = x + C_i'$ is upper hemi-continuous. By the closed graph theorem, Γ is upper hemicontinuous, and so is $\Gamma \cap \Gamma_i$. Hence, $(\Gamma \cap \Gamma_i)^u(\emptyset) = \{x \in K : \Gamma(x) \cap \Gamma_i(x) = \emptyset\}$ is open. Define $F_i := \mathbb{R}^d \setminus (\Gamma \cap \Gamma_i)^u(\emptyset) = \mathbb{R}^d \setminus K \cup \{x \in K : \Gamma(x) \cap \Gamma_i(x) \neq \emptyset\}$, which is closed by complementation. Since $K \subseteq \Gamma_0(x) \cup \cdots \cup \Gamma_d(x)$ for each $x \in K$, we always have $\Gamma(x) \cap \Gamma_i(x) \neq \emptyset$ for some i, hence the F_i's form a closed covering of K. Moreover $a_i \in F_i$ for each i since $\Gamma(a_i) \subseteq K \subseteq \Gamma_i(a_i)$. If F_i meets the facet S^i_{d-1} of K opposite to a_i, there exists $y \in S^i_{d-1}$ such that $\Gamma(y) \cap \Gamma_i(y) \neq \emptyset$. This means that $y = \Gamma(y)$. Suppose now that F_i does not meet S^i_{d-1}. Then F_i does not meet any faces of K contained in S^i_{d-1}. That is, $\operatorname{conv}\{a_j : j \in I\} \cap F_i = \emptyset$ for every $I \subseteq \{0, 1, \dots, d\} \setminus \{i\}$. But since the F_i's cover K, we necessarily have $\operatorname{conv}\{a_j : j \in I\} \subseteq \bigcup_{j \in I} F_j$ for every $I \subseteq \{0, 1, \dots, d\} \setminus \{i\}$. Therefore, the assumptions of the KKM lemma are satisfied, and so $\bigcap_{i=0}^d F_i \neq \emptyset$. Let $y \in \bigcap_{i=0}^d F_i \neq \emptyset$. Then $\Gamma(y)$ meets all the $\Gamma_i(y)$, and so $y = \Gamma(y)$.

(2) Let $S^{(k)}$ be the barycentric division of order k of K. For each vertex a^k of $S^{(k)}$, let $b^k \in \Gamma(a^k)$ and write $b^k = \phi_k(a^k)$. For each $x \in K$, we consider a d-simplex $\operatorname{conv}(\{a^k_{i_0}, a^k_{i_1}, \dots, a^k_{i_d}\}) \in S^{(k)}$ that contains x and define the function ϕ_k by $\phi_k(x) = p_0 \phi_k(a_{i_0}) + \cdots + p_d \phi_k(a_{i_d})$ where $x = p_0 a_{i_0} + \cdots + p_d a_{i_d}$. The function so defined is linear in the interior of a simplex of $S^{(k)}$ and so is continuous. Moreover, it is uniquely defined (even at points which belong to several d-simplices of $S^{(k)}$), hence ϕ_k is continuous on K. Therefore, by part (1) of the proof, there exists $x_k \in K$ such that $x_k = \phi_k(x_k)$. Since K is compact, the sequence $(x_k)_{k \in \mathbb{N}}$ has a limit point x_0. We now prove that $x_0 \in \Gamma(x_0)$. If $\operatorname{conv}(\{a^k_0, a^k_1, \dots, a^k_d\}) \in S^{(k)}$ contains x_k, then $x_k = \phi_k(x_k) = \sum_{i=0}^d p^k_i b^k_i$ where $x = \sum_{i=0}^d p^k_i a^k_i$. Let (x_{k_m}) be a subsequence of (x_k) that converges to x_0 and (l_m) a subsequence of (k_m) such that $b^{l_m}_i \to b^0_i$ and $p^{l_m}_i \to p^0_i$ for $i = 0, 1, \dots, d$. We have $p^0_i \geq 0$, $\sum_{i=0}^d p^0_i = 1$, and $\sum_{i=0}^d p^0_i b^0_i = x_0$. Since $b^{l_m}_i \in \Gamma(a^{l_m}_i)$ and Γ is upper hemicontinuous, we have $b^0_i \in \Gamma(x_0)$. Since $\Gamma(x_0)$ is convex, we have $x_0 \in \Gamma(x_0)$.

(3) Choose an interior point of K for origin 0 and consider an n-simplex S which contains K in its interior. Define the function ϕ by $\phi(x) = x$ if $x \in K$ and $\phi(x)$ equal to the extremity of $[0, x] \cap K$ if $x \in S \setminus K$. Then ϕ is continuous, and so $\Gamma \circ \phi$ is upper hemicontinuous. The conditions of part (2) of the proof apply, so there exists $x_0 \in S$ such that $x_0 \in \Gamma(\phi(x_0))$. Since $\Gamma(\phi(x_0)) \subseteq K$ by definition, we have $x_0 \in K$, and so $x_0 = \phi(x_0)$. Therefore, $x_0 \in \Gamma(\phi(x_0)) = \Gamma(x_0)$. □

Reference. This is exactly the proof of Berge (1963), which is not very different from the initial proof of Kakutani (1941). Another proof due to Cellina (1969) is based on selection theorems which allow to find a continuous function whose graph is within the graph of some correspondence.

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