Lecture 13. Riesz–Markov–Kakutani representation theorem

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Last updated: 31 December, 2023

Theorem 1 (**Riesz–Markov–Kakutani Representation Theorem** (for Compact Metric Spaces))**.** *Let X be a compact metric space and $\varphi \in C(X)'$ a continuous linear functional on $C(X)$. Then there exists a unique signed finite Borel measure $\mu \in \mathcal{M}_f^{\pm}(X)$ on X such that for all $f \in C(X)$,*

$$\varphi(f) = \int_X f(x)\, d\mu(x).$$

In this case, it holds that $\|\varphi\| = |\mu|(X)$. In particular, $C(X)$ and $\mathcal{M}_f^{\pm}(X)$ are isometrically isomorphic.

Proof. The proof is divided in three steps. Step 1 will prove existence. Step 2 will prove uniqueness. Step 3 will prove that $\|\varphi\| = |\mu|(X)$. Note that since X is compact, the space $C(X)$ can be normed, and so there is equivalence between boundedness and continuity for functionals in the dual $C(X)'$.

Step 1. Write $\varphi = L - L'$ by the Jordan decomposition for bounded functionals where L and L' are positive bounded functionals. Then apply the Riesz representation theorem for positive linear functionals to get positive finite measures μ^+ and μ^- such that $L(f) = \int f\, d\mu^+$ and $L'(f) = \int f\, d\mu^-$. Then $\mu = \mu^+ - \mu^-$ is a finite signed measure such that $\varphi(f) = \int f\, d\mu$ for all $f \in C(X)$.

Step 2. Suppose that there are $\mu_1, \mu_2 \in \mathcal{M}_f^{\pm}$ such that $\int f\, d\mu_1 = \int f\, d\mu_2$ for all $f \in C(X)$. To show that $\mu_1 = \mu_2$, it is enough to show that $\mu_1(F) = \mu_2(F)$ for all closed sets $F \subseteq X$. To show this, it is enough to show for each F, there exists a sequence of nonnegative functions $\{f_n\} \subseteq C(X)$ such that $f_n \to \mathbb{1}_F$ as $n \to \infty$. Define $h_n(r) = 1 - nr$ for $0 \le r \le 1/n$ and $h_n(r) = 0$ for $r > 1/n$. Then $f_n(x) = h_n(d(x, F))$ is continuous with $f_n \to \mathbb{1}_F$ as $n \to \infty$. Then $\mu_1(F) = \mu_2(F)$ follows from Lebesgue's dominated convergence and the uniqueness of limits.

Step 3. By the triangle inequality, we always have $\|\varphi\| \le \|L\| + \|L'\| = L(1) + L'(1) = |\mu|(X)$. To prove the reverse inequality, take any function $f \in C(X)$ such that $0 \le f \le 1$. Then $\|2\phi - 1\| \le 1$, and so $\|\varphi\| \ge 2\varphi(f) - \varphi(1)$. Taking supremum over all such f and remembering the definition of L, we get $\|\varphi\| \ge 2L(1) - \varphi(1) = L(1) + L'(1) = |\mu|(X)$. This concludes the proof. □

Reference. T.RRTDC in Royden RA p.464. The first part is also in T.17.8. in Bass RAGS p.180. While uniqueness follows from T.1.2. in Billingsley CPM p.8. The norm equality follows from P.12. in Royden RA p.463. For a similar proof of a slightly more general result, see T.7.2. and T.7.17. in Folland RA p.212-223 or T.12.9.-10. in Teschl TRFA p.363-364. A similar proof is T.2.14. in Rudin RCA p.40, but the proof is convoluted as it hides inside some form of Carathéodory's extension theorem. Another standard proof is through the Daniell integral (see R.7.2. in Folland RA p.232) – see T.10.4. in Bogachev MT2 p.111 or T.7.4.1. in Dudley RAP p.239.

Lemma 2 (Carathéodory's (Extension) Theorem). *If μ^* is an outer measure on X, then the collection $\mathcal{A}$ of μ^*-measurable sets is a σ-algebra and the restriction of μ^* to $\mathcal{A}$ is a complete measure.*

Proof. Recall that a set $A \subseteq X$ is said to be μ^*-measurable for μ^* an outer measure if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $E \subseteq X$. By definition of an outer measure, $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, hence μ^*-measurability more simply holds if $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

We first show that $\mathcal{A}$ is an algebra. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ by symmetry of μ^*-measurability with respect to A and A^c. Suppose that $A, B \in \mathcal{A}$ and $E \subseteq X$, then

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu * (E \cap A \cap B) + \mu * (E \cap A \cap B^c) + \mu * (E \cap A^c \cap B) + \mu * (E \cap A^c \cap B^c).\end{aligned}$$

Since $A \cup B \subseteq (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$ and $A^c \cap B^c = (A \cup B)^c$, then

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

This shows that $A \cup B \in \mathcal{A}$, and thus $\mathcal{A}$ is an algebra.

We now show that $\mathcal{A}$ is a σ-algebra. Let A_i be pairwise disjoint sets in $\mathcal{A}$, $B_n = \bigcup_{i=1}^n A_i$, and $B = \bigcup_{i=1}^\infty A_i$. If $E \subseteq X$, then

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}).\end{aligned}$$

Since $\mu^*(E \cap B_{n-1}) = \mu^*(E \cap A_n n - 1) + \mu^*(E \cap B_{n-1})$, we obtain recursively that

$$\mu^*(E \cap B_n) \geq \sum_{i=1}^n \mu^*(E \cap A_i).$$

Since $B_n \in \mathcal{A}$, we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_N^c) \geq \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B_n^c).$$

Taking $n \to \infty$, we obtain that

$$\begin{aligned}
\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c) \\
&\geq \mu^*(\bigcup_{i=1}^{\infty} E \cap A_i) + \mu^*(E \cap B^c) \\
&= \mu^*(E \cap B) + \mu^*(E \cap B^c) \\
&\geq \mu^*(E).
\end{aligned}$$

Therefore $B \in \mathcal{A}$. Now, take $C_1, C_2, \cdots \in \mathcal{A}$ and define $D_1 = A_1$ and for any $i \geq 2$, $D_i = C_i - \bigcup_{j=1}^{i-1} D_j$. Since $C_i \in \mathcal{A}$ and $\mathcal{A}$ is an algebra, we have that $D_i = C_i \cap (C_1 \cup \cdots \cup C_{i-1})^c \in \mathcal{A}$. Moreover, the D_i are pairwise disjoint, so it follows from the previous result that

$$\bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} D_i \in \mathcal{A}$$

and $\bigcap_{i=1}^{\infty} C_i = (\bigcup_{i=1}^{\infty} C_i^c)^c \in \mathcal{A}$, so $\mathcal{A}$ is a σ-algebra.

We now show that μ^* restricted to $\mathcal{A}$ is a measure. From a previous display, we showed that

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c).$$

Taking $E = B$, we obtain

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

This proves that μ^* is countably additive on $\mathcal{A}$.

We finally show that this restriction is complete. If $\mu^*(A) = 0$ and $E \subseteq X$, then

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leq \mu^*(E),$$

and so $A \in \mathcal{A}$, which proves that the restriction of μ^* to $\mathcal{A}$ is complete. □

Reference. T.1.11. in Folland RA p.29 or T.4.6. in Bass RAGS p.26. The naming conventions differ: the theorem as stated is the main ingredient of what is usually known as Caratheodory's extension theorem which starts from premeasure on rings.

Lemma 3 (Urysohn's Lemma for Metric Spaces)**.** *Let A and B be two disjoint closed subsets of a metric space X. Then there exists a continuous real-valued function $f : X \to$* $[0, 1]$ *such that $f = 0$ on A and $f = 1$ on B.*

Proof. If d denotes the metric on X, then the function

$$f(x) = \frac{d(x, A)}{d(x, B) + d(x, A)}$$

is continuous, with values in $[0, 1]$, and satisfies $f = 0$ on A and $f = 1$ on B. □

Reference. E.B.20. in Teschl TRFA p.541.

Lemma 4 (Partition of Unity). *Let X be a compact metric space. If K is a compact subset of X and $G_1, G_2, \dots, G_n$ is an open cover of K, then there exist continuous functions $g_i\colon X \to [0,1]$ with support contained in G_i, for $i = 1, 2, \dots, n$, such that*

$$\sum_{i=1}^{n} g_i(x) = 1$$

for all $x \in K$.

Proof. Let $x \in K$. There is $i \in \{1, 2, \dots, n\}$ such that $x \in G_i$. Singletons are closed, so, by Urysohn's lemma, there exists a continuous function $h_x\colon X \to [0,1]$ with support contained in G_i such that $h_x(x) = 1$. Define $N_x = \{y \in X : h_x(y) > 0\}$. Since h_x is continuous, then N_x is open, $x \in N_x$, and $\overline{N}_x \subseteq G_i$. The collection $\{N_x : x \in K\}$ is an open cover for the compact set K, so there exists a finite open subcover $\{N_{x_1}, \dots, N_{x_m}\}$ of K. For each $i \in \{1, 2, \dots, n\}$, define $F_i = \bigcup\{\overline{N}_{x_j} : \overline{N}_{x_j} \subseteq G_i\}$. Each F_i is closed, and so compact since X is compact. By definition, $F_i \subseteq G_i$. By Urysohn's lemma, we can choose continuous functions $f_i\colon X \to [0,1]$ with support contained in G_i such that $f_i = 1$ on F_i. Now define

$$\begin{aligned}
&g_1 = f_1,\\
&g_2 = (1 - f_1)f_2,\\
&\dots\\
&g_n = (1 - f_1)(1 - f_2)\dots(1 - f_{n-1})f_n.
\end{aligned}$$

By definition, g_i is a continuous function, with values in $[0,1]$, and support contained in G_i. Moreover, $g_1 + g_2 = 1 - (1 - f_1)(1 - f_2)$, and, by induction,

$$g_1 + g_2 + \dots + g_n = 1 - (1 - f_1)(1 - f_2)\dots(1 - f_n).$$

If $x \in K$, then $x \in N_{x_j}$ for some j, so $x \in F_i$ for some i. Then $f_i(x) = 1$, which implies that $\sum_{i=1}^{n} g_k(x) = 1$. □

Reference. P.17.2. in Bass RAGS p.172.

Lemma 5 (Riesz Representation Theorem for Positive Functionals). *Let X be a compact metric space and L a positive linear functional on $C(X)$. Then there exists a Borel measure $\mu \in \mathcal{M}_f(X)$ on X such that for all $f \in C(X)$,*

$$L(f) = \int_X f(x)\, d\mu(x).$$

Proof. Define for any open set $G \subseteq X$,

$$\mathcal{F}_G = \{f \in C(X) : 0 \le f \le 1, \operatorname{supp}(f) \subseteq G\},$$

$$l(G) = \sup\{L(f) : f \in \mathcal{F}_G\},$$

and for any set $E \subseteq X$,

$$\mu^*(E) = \inf\{l(G) : E \subseteq G,\ G \text{ open}\}.$$

The proof is divided in 5 steps. Step 1 will show that μ^* is an outer measure. Step 2 will show that every open set is μ^*-measurable. Step 3 will be an application of the Caratheodory extension theorem to obtain a measure μ from μ^*. Step 4 will establish some regularity properties of μ. Step 5 will show that the equality in the display of the theorem holds using the regularity properties established in Step 4.

Step 1. We show that μ^* is an outer measure. The set $\mathcal{F}_\emptyset$ only contains the zero function, so $l(\emptyset) = 0$, and thus $\mu^*(0) = 0$. If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$ by definition. Now, let $G_1, G_2, \ldots$ be open sets. For any open set H, $l(H) = \mu^*(H)$. Let $G = \bigcup_i G_i$ and let f be any element of $\mathcal{F}_G$ with support denoted by K. Since X is compact, K is compact since closed. Moreover, $\{G_i\}$ is an open cover for K, so there exists n such that $K \subseteq \bigcup_{i=1}^n G_i$. Let $\{g_i\}$ be a partition of unity for K subordinate to $\{G_i\}_{i=1}^n$. Since K is the support of f, we have $f = \sum_{i=1}^n f g_i$. Since $g_i \in \mathcal{F}_{G_i}$ and $f \leq 1$, then $f g_i \in \mathcal{F}_{G_i}$. Therefore,

$$L(f) = \sum_{i=1}^{n} L(f g_i) \leq \sum_{i=1}^{n} \mu^*(G_i) \leq \sum_{i=1}^{n} \mu^*(G_i).$$

Taking the supremum over $f \in \mathcal{F}_G$,

$$\mu^*(G) = l(G) \leq \sum_{i=1}^{\infty} \mu^*(G_i).$$

Finally, let $A_1, A_2, \ldots$ be arbitrary subsets of X. Let $\varepsilon > 0$. Choose G_i open such that $l(G_i) \leq \mu^*(A_i) + \varepsilon^{-i}$. Then

$$\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \mu^*(\bigcup_{i=1}^{\infty} G_i) \leq \sum_{i=1}^{\infty} \mu^*(G_i) \sum_{i=1}^{\infty} \mu^*(A_i) + \varepsilon.$$

Since ε is arbitrary, we have that μ^* is countably subadditive. This proves that μ^* is an outer measure.

Step 2. We show that every open set is μ^*-measurable. Suppose G is open and $E \subseteq X$. It suffices to show that $\mu^*(E) \geq \mu^*(E \cap G) + \mu^*(E \cap G^c)$. First suppose that E is open. Take $f \in \mathcal{F}_{E\cap G}$ such that $L(f) > l(E \cap G) - \varepsilon/2$. Denote K the support of f. Since K^c is open, we can choose $g \in \mathcal{F}_{E\cap K^c}$ such that $L(g) > l(E \cap K^c) - \varepsilon/2$. Then $f + g \in \mathcal{F}_E$ and

$$\begin{aligned}
l(E) &\geq L(f + g) \\
&\leq l(E \cap G) + l(E \cap K^c) - \varepsilon \\
&= \mu^*(E \cap G) + \mu^*(E \cap K^c) - \varepsilon \\
&\geq \mu^*(E \cap G) + \mu^*(E \cap G^c) - \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we have $\mu^*(E) \geq \mu^*(E \cap G) + \mu^*(E \cap G^c)$ when E is open. Suppose now that E is not open. Let $\varepsilon > 0$. Take H open such that $E \subseteq H$ and $l(H) < \mu^*(E) + \varepsilon$.

Then

$$\begin{aligned}\mu^*(E) + \varepsilon &\geq l(H) \\ &= \mu^*(H) \\ &\geq \mu^*(H \cap G) + \mu^*(H \cap G^c) \\ &\geq \mu^*(E \cap G) + \mu^*(E \cap G^c).\end{aligned}$$

Since ε was arbitrary, we have $\mu^*(E) \geq \mu^*(E \cap G) + \mu^*(E \cap G^c)$. This concludes that all open sets are μ^*-measurable.

Step 3. By Caratheodory's extension theorem, the restriction of μ^* to the Borel σ-algebra $\mathcal{B}$ on X is a measure on $\mathcal{B}$. By definition, if G is open, $\mu(G) = \mu^*(G) = l(G)$.

Step 4. We show that if K is compact, $f \in C(X)$, and $f \geq \mathbb{1}_K$, then $L(f) \geq \mu(K)$. Let $\varepsilon > 0$ and define

$$G = \{x \in X : f(x) > 1 - \varepsilon\},$$

which is open. Take $g \in \mathcal{F}_G$. Then $g \leq \mathbb{1}_K \leq f/(1-\varepsilon)$, so $(1-\varepsilon)^{-1} f - g \geq 0$. Because L is a positive linear functional, $L((1-\varepsilon)^{-1} f - g) \geq 0$. Thus $L(g) \leq L(f)/(1-\varepsilon)$. This is true for all $g \in \mathcal{F}_G$, then $\mu(G) \leq L(f)/(1-\varepsilon)$. Since $f \geq \mathbb{1}_K$, $\mu(K) \leq \mu(G)$. It follows that $\mu(K) \leq L(f)/(1-\varepsilon)$. Since ε was arbitrary, $\mu(K) \leq L(f)$.

Step 5. We finally show that $L(f) = \int_X f(x)\, d\mu(x)$. Without loss of generality, we can take $0 \leq f \leq 1$. Indeed, by writing $f = f^+ - f^-$ and using linearity of L, we may suppose $f \geq 0$. Moreover, since X is compact, f is bounded, so we can multiply by a constant and use linearity allowing us to take $f \leq 1$. Now, let $n \geq 1$ and $K_i = \{x \in X : f(x) \geq i/n\}$. Since f is continuous, each K_i is closed, and hence compact since X is compact. Note that $K_0 = X$. Define

$$f_i(x) = \begin{cases} 0, & \text{if } x \in K_{i-1}^c, \\ f(x) - \frac{i-1}{n} & \text{if } x \in K_{i-1} - K_i \\ \frac{1}{n} & \text{if } x \in K_i. \end{cases}$$

Note that $f = \sum_{i=1}^n f_i$ and $\mathbb{1}_{K_i} \leq n f_i \leq \mathbb{1}_{K_{i-1}}$. Therefore

$$\frac{1}{n}\mu(K_i) \leq \int f_i\, d\mu \leq \frac{1}{n}\mu(K_{i-1}),$$

and so

$$\frac{1}{n}\sum_{i=1}^{n} \mu(K_i) \leq \int f\, d\mu \leq \frac{1}{n}\sum_{i=0}^{n-1} \mu(K_i).$$

Let $\varepsilon > 0$ and G be an open set containing K_{i-1} such that $\mu(G) < \mu(K_{i-1}) + \varepsilon$. Then $n f_i \in \mathcal{F}_G$, so

$$L(n f_i) \leq \mu(G) \leq \mu(K_{i-1}) + \varepsilon.$$

Since ε was arbitrary, $L(f_i) \leq \mu(K_{i-1})/n$. By Step 4, $L(n f_i) \geq \mu(K_i)$, and thus

$$\frac{1}{n}\sum_{i=1}^{n} \mu(K_i) \leq L(f) \leq \frac{1}{n}\sum_{i=0}^{n-1} \mu(K_i).$$

It follows that
$$\left|L(f) - \int f \, d\mu\right| \le \frac{1}{n}(\mu(K_0) - \mu(K_n)) \le \frac{\mu(X)}{n}.$$
Since $\mu(X) = L(1) < \infty$ and n was arbitrary, the equality is proved. □

Reference. T.17.3. in Bass RAGS p.174.

Lemma 6 (Jordan Decomposition for Bounded Functionals)**.** *Let X be a compact metric space. If I is a bounded linear functional on $C(X)$, then there exist positive bounded linear functionals J and J' such that $I = J - J'$.*

Proof. For $g \in C(X)$ with $g \ge 0$, define
$$J(g) = \sup\{I(f) : f \in C(X), 0 \le f \le g\}.$$
Since $I(0) = 0$, then $J(g) \ge 0$. Since $|I(f)| \le \|I\|\|f\| \le \|I\|\|g\|$ if $0 \le f \le g$, then $|J(g)| \le \|I\|\|g\|$. Moreover, it holds that $J(cg) = cJ(g)$ if $c \ge 0$. We now prove that $J(g_1 + g_2) = J(g_1) + J(g_2)$ if $g_1, g_2 \in C(X)$ with $g_1, g_2 \ge 0$. If $0 \le f_1 \le g_1$ and $0 \le f_2 \le g_2$ with $f_1, f_2, g_1, g_2 \in C(X)$, we have $0 \le f_1 + f_2 \le g_1 + g_2$, so $J(g_1 + g_2) \ge I(f_1 + f_2) = I(f_1) + I(f_2)$. By taking the supremum over all f_1 and all f_2, we get $J(g_1 + g_2) \ge J(g_1) + J(g_2)$. Now, suppose that $0 \le f \le g_1 + g_2$ with $f, g_1, g_2 \ge 0$ and $f, g_1, g_2 \in C(X)$. Define $f_1 = f \wedge g_1$ and $f_2 = f - f_1$. Note that $f_1, f_2 \in C(X)$. Since $f_1 \le f$, then $f_2 \ge 0$. If $f(x) \le g_1(x)$, we have $f(x) = f_1(x) \le f_1(x) + g_2(x)$. If $f(x) > g_1(x)$, we have $f(x) \le g_1(x) + g_2(x) = f_1(x) + g_2(x)$. Hence $f \le f_1 + g_2$, so $f_2 = f - f_1 \le g_2$. Thus $I(f) = I(f_1) + I(f_2) \le J(g_1) + J(g_2)$. Taking the supremum over $f \in C(X)$ with $0 \le f \le g_1 + g_2$, we get $J(g_1 + g_2) \le J(g_1) + J(g_2)$. It follows that $J(g_1 + g_2) \le J(g_1) + J(g_2)$.

Now define for any $f \in C(X)$ (not necessarily non-negative),
$$J(f) = J(f^+) - J(f^-).$$
Then additivity, homogeneity, positivity, and boundedness of J for f follow directly from that of J on $\mathbb{R}^+$ for f^+ and f^- proved above.

Finally, define $J' = J - I$. Additivity, homogeneity, and boundedness of J follows from that of J and I. For positivity, note that if $f \ge 0$, then $I(f) \le J(f)$. This concludes the proof. □

Reference. P.17.7. in Bass RAGS p.179 or L.7.15. in Folland RA p.221.

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